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Negative State Weighting in the Linear Quadratic Regulator for Aircraft Control

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I. Introduction

We develop a relation between the state-weighting matrix in the linear quadratic objective function and the open-loop and closed-loop eigenvalues of single-input systems. Thus, given the open-loop poles, the state weights that yield the desired closed-loop poles may be computed. This relation suggests that for some selections of the closed-loop poles negative state weighting is needed. Some illustrative examples are presented. A stability augmentation system is designed for the F-16 aircraft short-period approximation. In this design, negative state weighting is found necessary to achieve the MIL specifications.

Consider a linear system given by

$$\dot{x} = Ax + Bu \tag{1}$$

$$u = Kx \tag{2}$$

The linear quadratic (LQ) problem can be stated as follows. Find a feedback gain K so that the performance index

$$J = \int_0^\infty (x^TQx + u^TRu)\, dt \tag{3}$$

is minimized. The solution to this problem is well known.[1,10] The feedback gain K that solves the LQ problem is given by

$$K = -R^{-1}B^TP \tag{4}$$

where P is the solution to the algebraic matrix Riccati equation

$$A^TP + PA - PBR^{-1}B^TP + Q = 0 \tag{5}$$

To guarantee the existence of a unique solution, R is assumed to be positive definite ($R > 0$). A sufficient condition for the existence of a solution P for Eq. (5) is that Q must be positive semidefinite. However, it has been shown[2] that Q need not satisfy this condition for a solution to exist.

Some research has been reported on the LQ problem with nonpositive semidefinite Q. Shin and Chen[3] showed that a nonpositive semidefinite Q is needed to design a PID compensator for a second-order system using the Ziegler-Nichols design method. By solving the inverse LQ problem, Molinari[2] and Jameson and Kreinder[4] showed that for a given K satisfying certain conditions, K is shown to be a solution of an LQ problem with Q not necessarily positive semidefinite. Ohta and Kakinuma[5] developed a relation between the coefficients of the open- and closed-loop characteristic polynomial and the diagonal elements of the Q matrix for single-input/single-output systems (SISO) in controllable form. They also found that if the Kessler polynomial is used to describe desirable pole locations, some of the diagonal entries in the Q matrix may have to be negative. The negative weighting will result in a closed-loop transfer function that does not satisfy the circle condition (the magnitude of $[I + K(sI - A)^{-1}B]$ is greater than one) but may increase the high-frequency roll-off of the closed-loop systems, which is often advantageous. We recall that the usual LQ regulator with positive semidefinite Q always has a high-frequency roll-off of -20 dB/decade.[1]

In view of these results, it is very clear that restricting Q to be positive semidefinite will respect the region of allowable closed-loop poles. We would, therefore, like to look at some situations in which we have to use negative state weighting to achieve the required closed-loop pole locations.

In this paper we develop a relation between the open- and closed-loop poles and the weighting coefficients Q for reachable SISO systems. This relation is more convenient for design then the relation in Ref. 5 where the formulation is in terms of the coefficients of the characteristic polynomials. From this relation we can determine whether negative state weighting is needed by looking at the closed-loop and open-loop poles only. To accomplish this, we first transform the system to the controllable form. Some special cases of open-loop closed-loop pairs where positive semidefinite Q is not suitable will be presented. Finally, an aircraft stability augmentation system will be designed where negative weights are required to achieve the military performance specifications in Ref. 6.

II. LQ Weighting for Desired Closed-Loop Poles

Consider the system given by Eq. (1) where $x \in R^n$ and u is a scalar. Assuming (A,B) is controllable, this can be transformed to the controllable form[7]

$$\underline{\dot{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \\ -a_1 & -a_2 & -a_n \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \underline{u} \tag{6}$$

Let the performance index be as given in Eq. (3) with R equal to identify and Q a diagonal matrix. Suppose the open-loop characteristic equation is given by

$$P(s) = \prod_{i=1}^{n} (s - \lambda_i) \tag{7}$$

with eigenvalues given by

$$\lambda_i = a_i e^{j(\alpha_i)} \tag{8}$$

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and the desired closed-loop equation is given by $P_c(s)$:

$$P_c(s) = \prod_{i=1}^{n}(s - \lambda_{c_i}) \tag{9}$$

with eigenvalues given by

$$\lambda_{c_i} = b_i e^{j(\beta_i)} \tag{10}$$

Now, we have the following result.

Theorem. Select the state-weighting matrix of the LQ performance index as $Q = \mathrm{diag}\{q_i\}$, with

$$q_1 = \prod_{i=1}^{n}\lambda_{c_i}^2 - \prod_{i=1}^{n}\lambda_i^2 \tag{11}$$

$$q_i = \sum_{j_1=1}^{i}\sum_{j_2=j_1+1}^{i+1}\cdots\sum_{j_i=n-l+1}^{n}\left[\prod_{j=j_1}^{j_i}\lambda_{c_j}^2 - \prod_{j=j_1}^{j_i}\lambda_j^2\right] \qquad i = 2, \ldots, n-1 \tag{12}$$

and

$$q_n = \sum_{i=1}^{n}[\lambda_{c_i}^2 - \lambda_i^2] \tag{13}$$

Then the optimal gain of Eq. (4) results in the closed-loop poles of Eq. (10).

Proof. The optimal controller gain K that minimizes Eq. (3) satisfies[5]

$$P_c(s)P_c(-s) - P(s)P(-s) = B^T H(-s)^T Q H(s) B \tag{14}$$

where $H(s)$ is the adjoint matrix of $(sI - A)$ and is of the form

$$H(s) = \left[\begin{array}{cc|c} x & x & 1 \\ x & x & s \\ \cdot & \cdot & s^2 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ x & x & s^{n-1} \end{array}\right] \tag{15}$$

with x representing nonzero elements of no concern. To prove the result in the theorem, substitute for $P(s)$, $P_c(s)$, and $H(s)$ in Eq. (14). Now we have two polynomials in s^2 of order $(n - 1)$. The coefficients of the left side polynomial are functions of the open- and closed-loop poles of the system only. The coefficients of the right side polynomial are functions of q_i. Equating the coefficients of the powers of s yields Eqs. (11–13).

Equations (11–13) give a direct relation between the location of the open- and closed-loop poles and the required state weighting Q in the LQ performance index. Therefore, given the open-loop poles one may compute the Q required so that the LQ regulator yields desired closed-loop poles. A similar result was derived in Ref. 8, but evidently only for open-loop stable systems.

The result shows that in controllable form there is no loss of generality in assuming that $R = I$ and Q is diagonal, for Eqs. (11–13) allow one to place any desired poles. Although we started the derivation with systems in controllable form, systems not in this form can be handled as well. Let $x = T\underline{x}$ be the transformation used to transform the original system with state x to the controllable form. If we assume a diagonal Q in controllable form, the state weighting for the original system is given by T^TQT^{-1} that is not necessarily diagonal. Note that if Q has some negative diagonal elements, then T^TQT^{-1} will not be positive semidefinite.

III. Examples

In this section some obvious cases are presented in which negative weighting is needed for a second-order system. Before we proceed, let us rewrite the open-loop and the closed-loop poles in terms of the angle with the negative real axes as

$$\lambda_i = a_i e^{j(\pi - \underline{\alpha}_i)} \tag{16}$$

and

$$\lambda_{c_i} = b_i e^{j(\pi - \underline{\beta}_i)} \tag{17}$$

Using this representation, Eqs. (11–13) show that the state-weighting matrix for a second-order system in controllable form is diagonal with

$$q_1 = b_1^2 b_2^2 - a_1^2 a_2^2 \tag{18}$$

and

$$q_2 = b_1^2\cos(2\underline{\beta}_1) + b_2^2\cos(2\underline{\beta}_2) - a_1^2\cos(2\underline{\alpha}_1) - a_2^2\cos(2\underline{\alpha}_2) \tag{19}$$

Now we present some examples in which negative state weighting is needed.

Example 1

In this example we consider an open-loop system with real poles. The desired closed-loop poles are real and selected such that the magnitude of each of the closed-loop poles is less than the magnitude of each of the open-loop poles. From Eqs. (18) and (19) we have both q_1 and q_2 are negative.

Example 2

Here we consider a controller design that decreases the damping of the system. Assume that $\underline{\alpha}_1$ and $\underline{\beta}_1$ are both greater than $\pi/4$ and we want $\underline{\beta}_1$ to be greater than $\underline{\alpha}_1$. Assume also that the magnitudes of the open-loop and the closed-loop poles are equal. In this case, q_1 is equal to 0 and q_2 has to be negative.

Example 3

Assume that $\underline{\alpha}_1$ is in the interval $[3\pi/4, \pi]$ and $\underline{\beta}_1$ in the interval $[\pi/4, \pi/2]$. Assume also that the magnitudes of both the open-loop and the closed-loop poles are equal. This design requires $q_1 = 0$ and q_2 to be negative.

IV. Design Example

In this section we demonstrate the need for nonpositive semidefinite state weighting even in common cases by designing a longitudinal stability augmentation system.

The system under consideration is the short-period approximation to the F-16 dynamics linearized about the nominal flight condition (speed: 502 ft/s, 0 altitude, level flight, 300 psf dynamic pressure, with the c.g. at 0.4).[9] Thus the basic aircraft states of interest are the pitch q and angle of attack α. The control input is the elevator deflection. For simplicity in demonstrating our point that nonpositive semidefinite Q may be needed even in common cases, we neglect the elevator actuator. Under these conditions, the system dynamics is given by Eq. (1) with

$$A = \begin{bmatrix} -1.0188 & 0.905284 \\ 4.0639 & -0.77013 \end{bmatrix} \tag{20}$$

and

$$B = \begin{bmatrix} -0.00212 \\ -0.16919 \end{bmatrix} \tag{21}$$

The eigenvalues of the open-loop system are $(-2.8166, 1.0276)$. Thus, the short-period mode has become two real poles since the c.g. is so far aft.

To apply our results to this system, we need to transform it to the controllable form. This can be done using the transformation $A_c = T^{-1}AT$ and $B_c = T^{-1}B$ where

$$A_c = \begin{bmatrix} 0 & 1 \\ 2.8944 & -1.7889 \end{bmatrix}, \qquad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{22}$$

and

$$T = \begin{bmatrix} -0.1548 & -0.0021 \\ -0.181 & -0.1692 \end{bmatrix} \tag{23}$$

The objective of the design is to locate the closed-loop poles at $(-2 \pm j1.9)$. These poles are suitable in terms of flying qualities.[6]

Equations (18) and (19) result in the state-weighting matrix

$$Q = \begin{bmatrix} 49.5347 & 0 \\ 0 & -8.2090 \end{bmatrix} \tag{24}$$

The solution of Eq. (5) is given by

$$P = W_1 W_2^{-1} \tag{25}$$

where $W = [W_{1T} W_{2T}]^T$ is the matrix of the eigenvectors corresponding to the stable eigenvalues of the Hamiltonian matrix

$$H = \begin{bmatrix} A_c & -B_c B_c^T \\ -Q & -A_c^T \end{bmatrix} \tag{26}$$

Therefore, we may compute

$$P = \begin{bmatrix} 35.697 & 10.5322 \\ 10.5322 & 2.2216 \end{bmatrix} \tag{27}$$

The required optimal LQ gain is given by

$$K = [10.5322 \quad 2.2216] \tag{28}$$

The optimal controller gain for the original system is given by

$$KT^{-1} = [-53.4699 \quad -12.4609] \tag{29}$$

and the state-weighting matrix for the original system is given by

$$T^{-T}QT^{-1} = \begin{bmatrix} 1725.4 & 0.3187 \\ 318.7 & -295 \end{bmatrix} \tag{30}$$

Note that, even for this simple design that represents a common sort of problem, a nonpositive semidefinite state weighting is needed to achieve the military specifications.

V. Conclusions

We have developed a relation for finding the state-weighting matrix Q in the linear quadratic performance index to yield specified closed-loop poles for single-input systems. This relation expresses Q in terms of the open-loop and closed-loop eigenvalues. The relation shows the need for nonpositive semidefinite state weighting for some selections of the closed-loop poles. This means that restricting our attention to a positive semidefinite state-weighting matrix does not allow arbitrary placement of the closed-loop poles. To show that nonpositive semidefinite Q can be needed in practical situations, we designed a longitudinal stability augmentation system where such a Q was needed to conform to the MIL-specified closed-loop poles.

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Improved Tracking of an Agile Target

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I. Introduction

As the term is commonly interpreted, tracking refers to estimating the current state of the target; e.g., position, velocity, etc., from a spatiotemporal observation.[1] Prediction of future target location can be made on the basis of this estimate. Even with the new generation of electro-optical (EO) sensors, tracking an agile target is a difficult task. While there is well-developed theory of linear (or essentially linear) estimation and prediction, quite useful in benign environments, it is difficult to adapt the results to situations involving rapid changes. The models upon which the algorithms are based frequently do not exhibit the discrete maneuver regimes typical of a hostile encounter.

Synthesis of model-based tracking algorithm begins with the equations of motion of the target. Motion dynamics are conventionally rendered in terms of a linear Gauss Markov (LGM) model

$$\mathrm{d}x_t = Ax_t\,\mathrm{d}t + \mathrm{d}w_t \tag{1}$$

in which x_t is the state process, including the position, velocity, and in some instances, acceleration of the target, and $(\mathrm{d}/\mathrm{d}t)w_t$ is a wideband (white) process of intensity W [$W\mathrm{d}t = (\mathrm{d}w_t)(\mathrm{d}w_t)'$], selected to introduce uncertainty into the path of the target. It is traditionally assumed that the tracker avails itself of sensors that measure target motions,

$$\mathrm{d}y_t = Dx_t\,\mathrm{d}t + \mathrm{d}n_x \tag{2}$$

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